ON AN EXTREMAL PROPERTY OF CONSTRAINT REACTIONS

(OB ODNOM EKSTREMAL'NOM SVOISTVE REAKTSII SVIAZEI)

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B.Iu. KOGAN
(Moscow)

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According to Gauss' principle, the actual accelerations of a system of particles minimizes a certain function of the virtual accelerations. It will be shown below that an analogous assertion is valid with respect to the constraint reactions: the actual reactions minimize a certain function of the virtual reactions.

Consider a holonomic system, subject to the ideal constraints

$$f_k(x_1, \ldots, x_n, t) = 0 \qquad (k = 1, \ldots, p) \tag{1}$$

Let us form the sum

$$S = {}^1/_2 \sum m_i (\mathbf{w}_i - \mathbf{w}_i^\circ)^2 \tag{2}$$

where m_i is the mass of the ith particle, $\mathbf{w}_i$ is the accelaration imparted to it by the applied force $\mathbf{F}$ and the reaction $\mathbf{N}_i$, and $\mathbf{w}_i^\circ$ is the acceleration which this particle would have at the instant t if the system were not under the influence of the applied forces (for given positions and velocities of all particles of the system). The sum (2) depends upon the accelerations $\mathbf{w}_i$, and since these depend in turn, upon the forces $\mathbf{F}_i$ and the reactions $\mathbf{N}_i$, then, for given $\mathbf{F}_i$, the sum is a function of the $\mathbf{N}_i$. Let us investigate this functional dependence.

Let us denote the true reaction by $\mathbf{N}_i$, and use the symbol $\mathbf{N}_i'$ to denote arbitrary reactions which are compatible with the given constraints. More precisely, $\mathbf{N}_i'$ will denote arbitrary reactions which satisfy condition

$$\sum \mathbf{N}_i' \,\delta \mathbf{r}_i = 0 \tag{3}$$

where $\delta\mathbf{r}_i$ are virtual displacements compatible with the constraints (1). (Reactions which do not satisfy (3) are obviously not compatible with (1)). Let us now replace the actual reactions $\mathbf{N}_i$ by arbitrary reactions $\mathbf{N}_i'$ which satisfy (3). Then the accelerations $\mathbf{w}_i$ take on the values $\mathbf{w}_i' = (\mathbf{F}_i' + \mathbf{N}_i')/m_i$ (which are, in general, incompatible with the constraints), and the sum (2) receives an increment ΔS,

$$\Delta S = \sum m_i (\mathbf{w}_i - \mathbf{w}_i^\circ)\, \Delta\mathbf{w}_i + {}^1/_2 \sum m_i (\Delta\mathbf{w}_i)^2 \qquad (\Delta\mathbf{w}_i = \mathbf{w}_i' - \mathbf{w}_i) \tag{4}$$

Further, since the applied forces $\mathbf{F}_i$ are given, one has

$$m_i \Delta\mathbf{w}_i = \mathbf{N}_i' - \mathbf{N}_i = \Delta\mathbf{N}_i$$

and therefore

$$\Delta S = \sum (\mathbf{w}_i - \mathbf{w}_i^\circ)\, \Delta \mathbf{N}_i + {}^1/_2 \sum m_i (\Delta \mathbf{w}_i)^2 \tag{5}$$

Let us compute the first term of the right-hand side of (5). From (1) we obtain

$$\sum \frac{\partial f_k}{\partial x_j} x_j^{\cdot} + \frac{\partial f_k}{\partial t} = 0 \tag{6}$$

$$\sum \frac{\partial f_k}{\partial x_j} x_j^{\cdot\cdot} + \varphi (x_1, \ldots, x_n, x_1^{\cdot}, \ldots, x_n^{\cdot}, t) = 0 \qquad (k = 1, \ldots, p) \tag{7}$$

Consequently, for fixed $x_1, \ldots, x_n, x_1^{\cdot}, \ldots, x_n^{\cdot}, t$ we must have

$$\sum \frac{\partial f_k}{\partial x_j} (x_j^{\cdot\cdot} - x_j^{\cdot\cdot\circ}) = 0 \qquad (k = 1, \ldots, p) \tag{8}$$

where $x_j^{\cdot\cdot}$ and $x_j^{\cdot\cdot\circ}$ are the components of the accelerations $\mathbf{w}_i$ and $\mathbf{w}_i^\circ$. Comparing Equations (8) with Equations

$$\sum \frac{\partial f_k}{\partial x_j} \delta x_j = 0 \qquad (k = 1, \ldots, p) \tag{9}$$

which define the actual displacements (and keeping in mind that the δx_j satisfy (9), and thus also satisfy (3)), we arrive at the conclusion that

$$\sum \mathbf{N}_i' (\mathbf{w}_i - \mathbf{w}_i^\circ) = 0 \tag{10}$$

Thus, since (10) is valid for arbitrary reactions $\mathbf{N}_i'$ satisfying condition (3), and hence, in particular, it must also hold for the actual reactions $\mathbf{N}_i$, we have

$$\sum \mathbf{N}_i (\mathbf{w}_i - \mathbf{w}_i^\circ) = 0 \tag{11}$$

From (10) and (11) we now obtain that

$$\sum (\mathbf{w}_i - \mathbf{w}_i^\circ)\, \Delta \mathbf{N}_i = 0 \qquad (\Delta \mathbf{N}_i = \mathbf{N}_i' - \mathbf{N}_i) \tag{12}$$

From (12) and (5) we conclude that $\Delta S \geqslant 0$, that is, that the replacement of the reactions $\mathbf{N}_i$ by arbitrary virtual reactions $\mathbf{N}_i'$ leads to the increase of the sum (2). Consequently, if the sum (2) is regarded as a function of the virtual reactions, then in the case when these reactions coincide with the actual reactions, the sum considered has a minimum value.

If we repeat the considerations of the above proof in the case of impact, we obtain the following theorem: the impact impulses of the virtual reactions are such that they minimize the sum

$$S = {}^1/_2 \sum m_i (\mathbf{v}_i - \mathbf{v}_i^\circ)^2 \tag{13}$$

(the energy of the induced velocities). Here, $\mathbf{v}_i^\circ$ is the velocity before impact, and $\mathbf{v}_i$ is the velocity after impact, and the latter is considered as a function of the impact impulses of the virtual reactions (for given impact impulses of the acting forces).

Translated by J.B.D.